NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
25 University Place, New York 3, N. Y.

COPY NO.

IMM-NYU 227
FEBRUARY 1956

NEW YORK UNIVERSITY
INSTITUTE OF
MATHEMATICAL SCIENCES

A Potential Problem Connected With an Infinite Dock in a Running Stream

By ROBERT A. SPINELLI

PREPARED UNDER
CONTRACT Nonr-285(06)
WITH THE
OFFICE OF NAVAL RESEARCH

IMM-NYU 227
February 1956

A POTENTIAL PROBLEM CONNECTED WITH AN INFINITE DOCK IN A RUNNING STREAM

by

Robert A. Spinelli

This paper represents results obtained at the Institute of Mathematical Sciences, New York University, under the sponsorship of Contract Nonr-285(06), with the Office of Naval Research.

TABLE OF CONTENTS

Section		Page
1.	Introduction	1
2.	Formulation and Solution of Specific Potential Problem	4
3.	Determination of Behavior of Solution at 0 and ∞ . .	14
4.	Rigorous Proofs of Several of the Asymptotic Relations Obtained	31
	Bibliography	40

[1.1] In the paper, "The Motion of a Ship, as a Floating Rigid Body, in a Seaway", [1], J. J. Stoker and A. S. Peters set up a general perturbation scheme to obtain the motion of a ship moving with constant velocity into a sea of waves. The ship was assumed to be thin in the vertical plane, e.g. a knife cutting through water. All oscillations of the ship and the amplitude of the waves were assumed to be small. The problem was solved explicitly for a thin ship moving in a atraight line with the wave fronts perpendicular to the line of motion of the ship. The investigation which resulted in the solution of the specific potential problem contained herein was to consider a horizontally thin ship, or disc, skimming over the surface of the water into a sea of waves. The same perturbation scheme was used to set up the equations defining the motion, but the problem involved in actually solving for even a special case was seen to be quite different and, in general, more difficult. A brief resume is given here of the results obtained

[1.2] Let O - XYZ be a coordinate system fixed in space. OY is perpendicular to the surface of the water, which occupies the lower half-space, $Y \leq 0$. Let O - xyz be a moving coordinate system such that:

(1) O-xz plane coincides with O-XZ plane.
(2) Oy axis passes through the center of gravity (c.g.) of the ship.
(3) the ship moves along the OX axis.

Finally, let $O^1 - x^1y^1z^1$ be a coordinate system rigidly connected to the ship, such that:

(1) the O^1y^1 axis passes through the c.g. of the ship.
(2) the $O^1 - x^1y^1$ plane coincides with the midplane of the ship.

We then assume that the hull of the ship is given by the equation $y^1 = -\beta h(x^1, z^1)$, where β is the perturbation parameter.

We assume also that all six quantities determining the motion of the ship, the potential function, u, and the free surface, n, can all be expanded in powers of the parameter, β. We refer everything to the axis O - xyz. Then, substituting the series expansions in:

(1) the equation of motion of the center of gravity;

(2) the equation for the conservation of angular momentum;

and equating like powers of β, we get various equations which are then manipulated to yield the following:

(1) $$y_1 = \frac{1}{gA}\int_A (u_{1t} - s_o u_{1x})dA \ , \qquad A = \int_A dA$$

(2) $$\theta_{31} = \frac{1}{gB}\int_A x(u_{1t} - s_o u_{1x})dA \ , \qquad B = \int_A x^2 dA$$

(3) $$\theta_{11} = -\frac{1}{gC}\int_A z(u_{1t} - s_o u_{1x})dA \ , \qquad C = \int_A z^2 dA$$

(4) $$\frac{M}{\beta}\dot{s}_1 = \delta\int_A h_x(\chi_{1t} - s_o\chi_{1x})dA$$

(5) $$-\frac{M}{\beta}s_o\omega_1 = \delta\int_A h_z(\chi_{1t} - s_o\chi_{1x})dA$$

(6) $$\frac{I_{21}}{\beta}(\ddot{\theta}_{21} + \dot{\omega}_1) = \delta\int_A (zh_x - xh_z)(u_{1t} - s_o u_{1x})dA$$

where: A = projection on $0^1 - x^1z^1$ plane of the ship's hull (under water).

u_1 = first order term in expansion of potential, u.

θ_{i1} = first order term in expansion of angular displacements, θ_i.

s_1 = first order term in expansion of speed, s, of the ship.

ω_1 = first order term in expansion of angular rotation of 0 - xyz axes.

y_1 = first order term in expansion of vertical displacement of ship.

χ_1 = time dependent part of u_1, i.e. $u_1 = \chi_o + \chi_1$, where χ_o is time independent.

s_o = speed of ship (i.e. $s = s_o + \beta s_1 + \ldots$).

M = mass of ship.

δ = density of water.

g = gravitational constant.

I_{21} = moment of inertia of ship about $0y^1$ axis.

Next, by using (1) the kinematic condition at the hull and (2) the free surface condition, we derive the fact that the first-order potential function, u_1, must satisfy:

$$(1)\quad u_{1y} = -s_o(h_x - \theta_{31}) - \dot{y}_1 - \dot{\theta}_{31} + z\dot{\theta}_{11} \quad , \text{ on } y = 0 \text{ , in } A \text{ .}$$

$$(2)\quad gu_{1y} - 2s_o u_{1xt} + s_o^2 u_{1xx} + u_{1tt} = 0 \quad , \text{ on } y = 0 \text{ , outside } A \text{ .}$$

We have, therefore a potential problem that we must solve for u_1 in terms of the parameters θ_{31}, $\dot{\theta}_{31}$, $\dot{y}_1$, $\dot{\theta}_{11}$, which, in turn, depend upon u_1. u_1, of course, could have singularities at the edge of the disc, A. Thus we see that the above problem embodies difficulties which are considerable.

Due to the obvious difficulties involved in the general case discussed above, the problem is simplified to that of an infinite ship planing into calm water in two dimensions. This

simplification yields the added advantage of the introduction of complex variables. The time independent part, x_o, of the potential is then sought. This leads to a specific potential theory problem which is solved explicitly in a manner similar to the one used by A. S. Peters [2].

[2.1] Thus we seek a harmonic function, say $\psi(x,y)$, which satisfies:

Problem I:

(1) $\triangle\psi = 0, \quad y < 0$

(2) $k\psi_y + \psi_{xx} = 0, \; y = 0, \; x > 0, \; k = \frac{g}{s_o^2} > 0$

(3) $\psi_y = f(x), \; y = 0, \; x < 0$

(4) $\psi \longrightarrow 0, \; x^2 + y^2 \longrightarrow \infty$.

In this case, $f(x) = -s_o h_x$, no constant terms appearing as they do in the general three-dimensional case. This corresponds to the fact that an infinite ship cannot be tilted at any angle, however small, since it would lead to an infinite displacement at $-\infty$.

Now replace ψ_{xx} by $-\psi_{yy}$ in (2) and let $\psi_y = u$. The problem for u is:

Problem II:

(1) $\triangle u = 0, \quad y < 0$

(2) $ku - u_y = 0, \; y = 0, \; x > 0$

(3) $u = f(x), \; y = 0, \; x < 0$

(4) $u \longrightarrow 0, \; x^2 + y^2 \longrightarrow \infty$.

It becomes evident later on that $f(x)$ must be restricted so that it is integrable over $(-\infty, 0)$. In particular it will be assumed that

$$f(x) \sim \frac{1}{(-x)^{1+\varepsilon}}, \quad \varepsilon > 0, \quad \text{as} \quad x \longrightarrow -\infty .$$

We solve first $f(x) = \delta(x - a)$, $a < 0$, i.e.

Problem III: (1) $\triangle\tilde{u} = 0$ in $y < 0$

(2) $k\tilde{u} - \tilde{u}_y = 0$ on $y = 0$, $x > 0$

(3) $\tilde{u} = \delta(x-a)$ on $y = 0$, $x < 0$

(4) $\tilde{u} \longrightarrow 0$ as $x^2 + y^2 \longrightarrow \infty$.

Now let $\bar{u} = \tilde{u} + \frac{1}{\pi} \cdot \frac{y}{(x-a)^2+y^2} - \frac{1}{\pi} \cdot \frac{y}{x^2+y^2}$. The problem for $\bar{u}$ is then:

Problem IV: (1) $\triangle\bar{u} = 0$ in $y < 0$

(2) $k\bar{u} - \bar{u}_y = -\frac{1}{\pi} \cdot \frac{1}{(x-a)^2} + \frac{1}{\pi} \cdot \frac{1}{x^2}$ on $y = 0$, $x > 0$

(3) $\bar{u} = 0$ on $y = 0$, $x < 0$

(4) $\bar{u} \longrightarrow 0$ as $x^2 + y^2 \longrightarrow \infty$.

The reason for the term $\frac{1}{\pi} \cdot \frac{y}{x^2+y^2}$ will be evident later.

Finally, let $\bar{u} = \text{Re } F(z)$, where $F(z)$ analytic in $\text{Im } z < 0$. Thus we get, as the problem for $F(z)$, the following:

Problem V: (1) $F(z)$ analytic in $\text{Im } z < 0$

(2) $\text{Re}\left[kF(z) - iF^1(z) + \frac{1}{\pi} \cdot \frac{1}{(z-a)^2} - \frac{1}{\pi} \cdot \frac{1}{z^2}\right] = 0$

on $y = 0$, $x > 0$

(3) $\text{Re } F(z) = 0$ on $y = 0$, $x < 0$

(4) $F(z) \longrightarrow 0$ as $|z| \longrightarrow \infty$.

[2.2] We now solve for $F(z)$, using analytic continuation. Continuing $F(z)$ across $x > 0$, we have:

(1) $$kF(\bar{z}) - iF^1(\bar{z}) + \frac{1}{\pi} \cdot \frac{1}{(\bar{z}-a)^2} - \frac{1}{\pi} \cdot \frac{1}{z^2}$$

$$= -\left[\overline{kF(z) - iF^1(z) + \frac{1}{\pi} \cdot \frac{1}{(z-a)^2} - \frac{1}{\pi} \cdot \frac{1}{z^2}}\right]$$

(2) $F(ze^{-2\pi i}) = -\overline{F(\bar{z})}$. Substituting (2) into (1) there results:

$$(3)\quad -kF(ze^{-2\pi i}) - iF^{1}(ze^{-2\pi i}) + kF(z) - iF^{1}(z) + \frac{2}{\pi}\cdot\frac{1}{(z-a)^2} - \frac{2}{\pi}\cdot\frac{1}{z^2} = 0$$

which is a differential-difference equation for $F(z)$.

Now assume that $F(z)$ can be represented in form:

$F(z) = \int_{P(z)} e^{z\zeta} g(\zeta)\,d\zeta$ for $-\pi < \arg z < 0$ where $P(z)$ is a path, as yet undetermined, which may depend on z.

We also assume:

$$F(ze^{-2\pi i}) = \int_{P(ze^{-2\pi i})} e^{z\zeta} g(\zeta)\,d\zeta \quad \text{for} \quad -3\pi < \arg z < -2\pi$$

where $P(ze^{-2\pi i})$ is a path, as yet undetermined, which may depend on z.

Therefore

$$F^{1}(z) = \int_{P(z)} e^{z\zeta}\,\zeta g(\zeta)\,d\zeta$$

$$F^{1}(ze^{-2\pi i}) = \int_{P(ze^{-2\pi i})} e^{z\zeta}\,\zeta g(\zeta)\,d\zeta \quad .$$

Substituting in equation (3) we get:

$$\int_{P(ze^{-2\pi i})} (-k-i\zeta)g(\zeta)e^{z\zeta}\,d\zeta + \int_{P(z)} (k-i\zeta)g(\zeta)e^{z\zeta}\,d\zeta + \frac{2}{\pi}\frac{1}{(z-a)^2} - \frac{2}{\pi}\cdot\frac{1}{z^2} = 0 \ .$$

Let $\zeta e^{-2\pi i} = \zeta^1$ in first integral.

$$- \int_{e^{-2\pi i}P(ze^{-2\pi i})} (k+i\zeta^1)g(\zeta^1 e^{2\pi i})e^{z\zeta^1}d\zeta$$

$$+ \int_{P(z)} (k-i\zeta)g(\zeta)e^{z\zeta}d\zeta + \frac{2}{\pi}\cdot\frac{1}{(z-a)^2} - \frac{2}{\pi}\cdot\frac{1}{z^2} = 0$$

$$(4) \quad - \int_{P_1(z)} (k+i\zeta)g(\zeta e^{2\pi i})e^{z\zeta}d\zeta + \int_{P(z)} (k-i\zeta)g(\zeta)e^{z\zeta}d\zeta$$

$$+\frac{2}{\pi}\cdot\frac{1}{(z-a)^2} - \frac{2}{\pi}\cdot\frac{1}{z^2} = 0$$

where $P_1(z)$ is the path $P(ze^{-2\pi i})$ rotated clockwise through angle 2π. Now let

$$\frac{2}{\pi}\cdot\frac{1}{(z-a)^2} - \frac{2}{\pi}\cdot\frac{1}{z^2} = \int_{P(z)} e^{z\zeta}h(\zeta)d\zeta \quad .$$

We can rewrite (4) as:

$$\int_{P(z)} e^{z\zeta}\left[(k-i\zeta)g(\zeta) - (k+i\zeta)g(\zeta e^{2\pi i}) + h(\zeta)\right]d\zeta$$

$$+\int_{P(z)} (k+i\zeta)g(\zeta e^{2\pi i})e^{z\zeta}d\zeta - \int_{P_1(z)} (k+i\zeta)g(\zeta e^{2\pi i})e^{z\zeta}d\zeta = 0 \quad .$$

The above equation will be satisfied if:

$$(5) \qquad (k - i\zeta)g(\zeta) - (k + i\zeta)g(\zeta e^{2\pi i}) + h(\zeta) = 0$$

$$(6) \quad \int_{P(z)} (k+i\zeta)g(\zeta e^{2\pi i})e^{z\zeta}d\zeta - \int_{P_1(z)} (k+i\zeta)g(\zeta e^{2\pi i})e^{z\zeta}d\zeta = 0 \quad .$$

Let $P(z)$ be the path from $-\infty$ to 0 along negative imaginary axis.

Therefore

$$h(\zeta) = -\frac{2}{\pi}\,\zeta(e^{-\alpha\zeta} - 1) \ .$$

Equation (5) then becomes:

$$(7)\quad (\zeta+ik)g(\zeta) + (\zeta-ik)g(\zeta e^{2\pi i}) - \frac{2i}{\pi}\,\zeta(e^{-\alpha\zeta} - 1) = 0 \ .$$

We now look for a solution of (7). In the end result we may neglect (6) entirely and verify the boundary conditions directly.

[2.3] To solve (7), we must first solve the homogeneous equation:

$$(8)\qquad (\zeta + ik)g(\zeta) + (\zeta-ik)g(\zeta e^{2\pi i}) = 0 \ .$$

Let $g(\zeta) = \zeta^{n-\frac{1}{2}}\,\psi(\zeta)$ where n is an arbitrary integer. Therefore

$$(\zeta+ik)\zeta^{n-\frac{1}{2}}\,\psi(\zeta) - (\zeta-ik)\zeta^{n-\frac{1}{2}}\,\psi(\zeta e^{2\pi i}) = 0 \ , \quad \text{or}$$

$$(9)\qquad (\zeta+ik)\psi(\zeta) - (\zeta-ik)\psi(\zeta e^{2\pi i}) = 0 \ .$$

Let $\tilde{\psi}(\zeta) = \psi(k\zeta)$. Thus $\tilde{\psi}$ satisfies

$$(10)\qquad (\zeta+ i)\tilde{\psi}(\zeta) - (\zeta - i)\tilde{\psi}(\zeta e^{2\pi i}) = 0 \ .$$

$$\frac{\tilde{\psi}(\zeta e^{2\pi i})}{\tilde{\psi}(\zeta)} = \frac{\zeta + i}{\zeta - i}$$

$$\ln \tilde{\psi}(\zeta e^{2\pi i}) - \ln \tilde{\psi}(\zeta) = \ln \frac{\zeta + i}{\zeta - i} \quad \text{let } H(\zeta) = \ln \tilde{\psi}(\zeta)$$

$$(11)\qquad H(\zeta e^{2\pi i}) - H(\zeta) = \ln \frac{\zeta + i}{\zeta - i} \ .$$

Now, $H(\zeta)$ can be represented in form:

$$H(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{0} \frac{1}{t-\zeta} \ln\frac{t+i}{t-i}\,dt$$ $H(\zeta)$ is a function

analytic in $-\pi < \arg\zeta < \pi$, which satisfies (11) at least when ζ approaches negative axis from below. Since $\ln\frac{t+i}{t-i} \longrightarrow 0$ as $t \longrightarrow \infty$, we can continue $H(\zeta)$ into other regions of the ζ-plane by moving the path of integration clockwise or counter-clockwise. Hence, due to the preservation of a functional relation by analytic continuation, we may say that (11) holds wherever $H(\zeta)$ is analytic. Thus $\tilde{\psi} = e^{H(\zeta)}$ satisfies (10).

[2.4] We now derive the major properties of $H(\zeta)$ and $\tilde{\psi}(\zeta)$.

(1) $H(\zeta)$ regular in whole plane, except perhaps negative axis.

$\tilde{\psi}(\zeta)$ regular in whole plane, except perhaps negative axis.

(2) For ζ real and > 0,

$H(\zeta)$ is real, since $\ln\frac{t+i}{t-i}$ is pure imaginary on real axis. Therefore $\tilde{\psi}(\zeta) > 0$.

(3) For ζ real and < 0, (ζ approaching negative axis from below):

$$H(\zeta) = -\frac{1}{2}\ln\frac{\zeta+i}{\zeta-i} + P.V.(\zeta) \text{ where } P.V.(\zeta) \text{ is real.}$$

Therefore $\tilde{\psi}(\zeta) = \sqrt{\frac{\zeta-i}{\zeta+i}}\, e^{P.V.(\zeta)}$. Therefore $(\zeta+i)\tilde{\psi}(\zeta) > 0$.

(4) As $\zeta \longrightarrow \infty$,

$$H(\zeta) = \frac{1}{\pi}\frac{\ln\zeta}{\zeta} + O(\frac{1}{\zeta})$$ (Peters [2] pp. 338-9).

Therefore $\tilde{\psi}(\zeta) = H\,\frac{1}{\pi}\frac{\ln\zeta}{\zeta} + O(\frac{1}{\zeta})$ for $-\pi < \arg\zeta < \pi$.

(5) As $\zeta \longrightarrow 0$,

$$H(\zeta) = -\frac{1}{2}\ln\zeta + O(\zeta \ln\zeta)$$ (Peters, ibid.)

Therefore

$$\tilde{\psi}(\zeta) = \zeta^{-\frac{1}{2}} + O(\zeta^{\frac{1}{2}}\ln\zeta)\ .$$

The corresponding properties for $\psi(\zeta) = \tilde{\psi}(\zeta/k)$ are:

(1) $\psi(\zeta)$ regular in whole plane except, perhaps, negative axis.

(2) For ζ real and > 0,

$$\psi(\zeta) > 0 \ .$$

(3) For ζ real and < 0,

$$(\zeta + ik)\psi(\zeta) > 0 \ .$$

(4) As $\zeta \longrightarrow \infty$, $\psi(\zeta) = 1 + \frac{k}{\pi} \frac{\ln \zeta}{\zeta} + O(\frac{1}{\zeta})$.

(5) As $\zeta \longrightarrow 0$, $\psi(\zeta) = k^{\frac{1}{2}} \zeta^{-\frac{1}{2}} + O(\zeta^{\frac{1}{2}} \ln \zeta)$.

[2.5] To the general solution of the homogeneous equation we must add a particular solution of the non-homogeneous equation. To get one such solution, let

$$g(\zeta) = g_o(\zeta)g_1(\zeta)$$

where $g_o(\zeta)$ is a particular solution, as yet undetermined, of the homogeneous equation, i.e.

$$(\zeta+ik)g_o(\zeta) + (\zeta-ik)g_o(\zeta e^{2\pi i}) = 0 \ .$$

Substituting in (7):

$$(\zeta+ik)g_o(\zeta)g_1(\zeta) + (\zeta-ik)g_o(\zeta e^{2\pi i})g_1(\zeta e^{2\pi i}) - \frac{2i}{\pi} \zeta(e^{-a\zeta} - 1) = 0$$

or

$$(\zeta+ik)g_o(\zeta)g_1(\zeta) - (\zeta+ik)g_o(\zeta)g_1(\zeta e^{2\pi i}) - \frac{2i}{\pi} \zeta(e^{-a\zeta} - 1) = 0 \ .$$

Therefore

$$g_1(\zeta) - g_1(\zeta e^{2\pi i}) = \frac{2i}{\pi} \cdot \frac{\zeta(e^{-a\zeta} - 1)}{(\zeta+ik)g_o(\zeta)} \ .$$

As we did for $H(\zeta)$ previously, we may write:

$$g_1(\zeta) = \frac{1}{2\pi i}\int_0^{-\infty} \frac{1}{t-\zeta}\left[\frac{2i}{\pi}\cdot\frac{t(e^{-dt}-1)}{(t+ik)g_o(t)}\right]dt \quad .$$

Now it is evident that $g_o(t)$ must be chosen so as to get a convergent integral for $g_1(\zeta)$.

Hence, finally, the general solution of (6) will be

$$c_1 g(\zeta) + c_2 g_o(\zeta) g_1(\zeta)$$

where c_1, c_2 are arbitrary constants and $g(\zeta)$, $g_o(\zeta)$ are solutions of (8). It will be seen that in order to (a) satisfy the boundary conditions and (b) restrict the singularity at the origin, there is only one choice for c_1, c_2 and $g_o(\zeta)$.

[2.6] We now verify that $F(\zeta)$ satisfies the boundary conditions for suitable choice of c_1, c_2 and any $g(\zeta)$, $g_o(\zeta)$.

$$F(z) = c_1\int_{-i\infty}^{0} e^{z\zeta} g(\zeta)d\zeta + c_2\int_{-i\infty}^{0} e^{z\zeta} g_o(\zeta)g_1(\zeta)d\zeta \quad .$$

I. Let $z = x < 0$.

$$F(x) = c_1\int_{-i\infty}^{0} e^{x\zeta} g(\zeta) + c_2\int_{-i\infty}^{0} e^{x\zeta} g_o(\zeta)g_1(\zeta)d\zeta \quad .$$

We may rotate the path into positive $\zeta = \xi$ axis.* I.e.

$$F(x) = c_1\int_{\infty}^{0} e^{x\xi} g(\xi)d\xi + c_2\int_{\infty}^{0} e^{x\xi} g_o(\xi)g_1(\xi)d\xi \quad .$$

---

* Actually in the rotating of the above path, we should first rotate the path into the positive axis and then let $y \longrightarrow 0$ for then Re $z\zeta = x\xi - yn < 0$ and $\longrightarrow -\infty$ on any ray in the lower-right quarter plane. The integrands die out exponentially at $|z| = \infty$.

Now since $g(\zeta) = \zeta^{n-\frac{1}{2}}\psi(\zeta)$, $g_o(\zeta) = \zeta^{m-\frac{1}{2}}\psi(\zeta)$ and from properties of $\psi(\zeta)$ we have:

(a) $g(\xi)$, $g_o(\xi)$ real on $\xi > 0$ axis.

(b) $(t+ik)g_o(t)$ pure imaginary on $t < 0$, hence $g_1(\xi)$ pure imaginary on $\xi > 0$ axis.

Hence if c_1 pure imaginary, c_2 real, we will have Re $F(x) = 0$, or boundary condition (3) is satisfied.

II. Let $z = x > 0$.

$$kF(x) - iF^1(x) = -ic_1 \int_{-i\infty}^{0} e^{x\zeta}(\zeta+ik)g(\zeta)d\zeta$$

$$- ic_2 \int_{-i\infty}^{0} e^{x\zeta}(\zeta+ik)g_o(\zeta)g_1(\zeta)d\zeta \quad .$$

Rotate path into negative $\zeta = \xi$. (cf. * above).

$$kF(x) - iF^1(x) = ic_1 \int_{-\infty}^{0} e^{x\xi}(\xi+ik)g(\xi)d\xi$$

$$- ic_2 \int_{-\infty}^{0} e^{x\xi}(\xi+ik)g_o(\xi)g_1(\xi)d\xi \quad .$$

Now,

(a) $(\xi+ik)g(\xi)$, $(\xi+ik)g_o(\xi)$ are pure imaginary on $\xi < 0$.

(b) For $\zeta = \xi < 0$,

$$g_1(\xi) = \frac{1}{\pi} \cdot \frac{\xi(e^{-a\xi} - 1)}{(\xi+ik)g_o(\xi)} + P.V.(\xi)$$

(c) $P.V.(\xi)$ is pure imaginary from (a) above. Hence,

$$\text{Re}\,[kF(x) - iF^{1}(x)] = \text{Re}\,\frac{c_2}{\pi}\int_{-\infty}^{0} e^{x\xi}\xi(e^{-a\xi} - 1)d\xi \quad \text{since } c_1 \text{ pure imag.}$$

c_2 real from I.

$$= -\frac{1}{\pi}\frac{1}{(x-a)^2} + \frac{1}{\pi}\cdot\frac{1}{x^2} \quad \text{if } c_2 = 1 .$$

Thus boundary condition (2) is satisfied. Therefore we see that for c_1 pure imaginary and $c_2 = 1$, $F(z)$ satisfies the proper boundary conditions for arbitrary $g(\zeta)$, $g_o(\zeta)$. $F(z)$ is thus a solution of problem V (with regard to 1), 2), 3) at least).

[2.7] It is a little more convenient to go back to the original ψ of problem I, before investigating behavior as $z \to 0$, $z \to \infty$. Thus,

$$\tilde{u} = \text{Re}\; c_1 \int_{-i\infty}^{0} e^{z\zeta} g(\zeta)d\zeta + \text{Re}\int_{-i\infty}^{0} e^{z\zeta} g_o(\zeta)g_1(\zeta)d\zeta$$

$$+\text{Re}\,-\frac{1}{\pi}\left[\frac{1}{z-a} - \frac{1}{z}\right]$$

is a solution of problem III. Integrating with respect to y from $(-\infty, y)$ we have

$$u = \text{Re}\; -ic_1 \int_{-i\infty}^{0} e^{z\zeta}\,\frac{g(\zeta)}{\zeta}\,d\zeta - \text{Re}\; i\int_{-i\infty}^{0} e^{z\zeta}\,\frac{g_o(\zeta)g_1(\zeta)}{\zeta}\,d\zeta$$

$$-\frac{1}{\pi}\,\text{Re}\,\log\frac{z-a}{z}$$

is a solution of problem I with $f(x) = \delta(x-a)$. [In this step we see the purpose of the seemingly extraneous term $\frac{1}{\pi}\cdot\frac{y}{x^2+y^2}$. Without it we would only have $\text{Re}\,-\frac{i}{\pi}\,\frac{1}{z-a}$ which is $O(\frac{1}{y})$ at $y = \infty$ and hence we would not be able to integrate directly to get u.]

Finally,

$$\psi = \mathrm{Re}\ -ic_1 A \int_{-i\infty}^{0} e^{z\zeta}\,\frac{g(\zeta)}{\zeta}\,d\zeta - \mathrm{Re}\ i \int_{-i\infty}^{0} e^{z\zeta}\,\frac{g_0(\zeta)\tilde{g}_1(\zeta)}{\zeta}\,d\zeta$$

$$- \frac{1}{2\pi}\int_{-\infty}^{0} f(\alpha)\,\log\frac{(x-\alpha)^2+y^2}{x^2+y^2}\,d\alpha$$

where

$$A = \int_{-\infty}^{0} f(\alpha)d\alpha\ ,\quad \tilde{g}_1(\zeta) =$$

$$= \frac{1}{2\pi i}\int_{0}^{-\infty} \frac{1}{t-\zeta}\left[\frac{2i}{\pi}\cdot\frac{t\int_{-\infty}^{0} f(\alpha)\cdot(e^{-\alpha t}-1)d\alpha}{(t+ik)g_0(t)}\right]dt$$

is a solution of problem I with $\psi_y = f(x)$, as desired.

[3.1] We now investigate the properties of ψ as $z \to 0$, $z \to \infty$. For convenience, let $\psi = \mathrm{Re}\ -ic_1 A\cdot I_1 - \mathrm{Re}\ i\ I_2 - \frac{1}{2\pi}\cdot I_3$ where

$$I_1 = \int_{-i\infty}^{0} e^{z\zeta}\,\frac{g(\zeta)}{\zeta}\,d\zeta\ ,\qquad I_2 = \int_{-i\infty}^{0} e^{z\zeta}\,\frac{g_0(\zeta)\tilde{g}_1(\zeta)}{\zeta}\,d\zeta\ ,$$

$$I_3 = \int_{-\infty}^{0} f(\alpha)\,\log\frac{(x-\alpha)^2+y^2}{x^2+y^2}\,d\alpha\ .$$

[3.2] In what follows, no attempt is made to derive asymptotic expressions rigorously, at first. Rather, a method will be used which indicates, but does not rigorously establish, what these expressions must be. In the end, several proofs will be given which show how the results may be verified. This is done to save space since every case involving the same type of integral needs slightly different considerations.

[3.3]

$$I_1 = \int_{-i\infty}^{0} e^{z\zeta} \frac{g(\zeta)}{\zeta} d\zeta \quad \text{where} \quad \frac{g(\zeta)}{\zeta} = \zeta^{m-\frac{1}{2}} \psi(\zeta) \quad ,$$

m an arbitrary integer.

From the properties of $\psi(\zeta)$ listed earlier, we have:

(1) $\zeta \longrightarrow \infty$, $\frac{g(\zeta)}{\zeta} = \zeta^{m-\frac{1}{2}} + \frac{k}{\pi} \zeta^{m-\frac{3}{2}} \ln \zeta + O(\zeta^{m-\frac{3}{2}})$

(2) $\zeta \longrightarrow 0$, $\frac{g(\zeta)}{\zeta} = k^{\frac{1}{2}} \zeta^{m-1} + O(\zeta^m \ln \zeta)$.

We obviously must have $m \geq 1$ for convergence of I_1 at $\zeta = 0$.

(a) Let $\zeta = \rho e^{i\alpha}$, $z = re^{i\theta}$, $-\frac{\pi}{2} \leq \theta < 0$, $r \longrightarrow \infty$.

Put $\alpha = -\pi-\theta$ so that $\theta+\alpha = -\pi$. In the range $-\pi-\theta \leq \alpha \leq -\frac{\pi}{2}$, we have that $-\frac{3\pi}{2} < -\frac{3\pi}{2} - \theta \leq \theta+\alpha < -\frac{\pi}{2}$ so that Re $z\zeta < 0$. Thus the shift of the path of integration is valid. Therefore

$$I_1 = \int_{\infty}^{0} e^{-r\rho} \frac{g(\rho e^{i\alpha})}{\rho} d\rho = \int_{\infty}^{0} e^{-\tau} \frac{g(\frac{\tau}{r} e^{i\alpha})}{\frac{\tau}{r}} \cdot \frac{d\tau}{r} \quad .$$

For $r \longrightarrow \infty$, I_1 depends only upon the behavior of g in a neighborhood of the origin. Therefore

$$I_1 \sim -\int_{0}^{\xi r} e^{-\tau} \left[k^{\frac{1}{2}} e^{im\alpha} \frac{\tau^m}{r^m} + \frac{1}{r^{m+1}} O(\tau^{m+1} \ln \tau) \right] \frac{d\tau}{\tau} \sim \frac{c}{r^m} \quad .$$

If $-\pi < \theta \leq -\frac{\pi}{2}$, let $\alpha = -\pi-\theta$. In the range $-\frac{\pi}{2} \leq \alpha \leq -\pi-\theta$, we have $-\frac{3\pi}{2} < \theta+\alpha \leq -\frac{3\pi}{2} - \theta < -\frac{\pi}{2}$ so that Re $z\zeta < 0$ and the same argument holds. For $\theta = 0$, as in the verification of the boundary conditions, we first shift the path and then put $\theta = 0$.

Thus as $|z| = r \longrightarrow \infty$,

$$I_1 \sim \frac{c}{r^m} .$$

(b) Let $r \longrightarrow 0$. We may consider only the path from any finite non-zero value of ρ to ∞ since, as $r \longrightarrow 0$, $\int_{\xi}^{0} \longrightarrow$ constant. Thus we consider

$$I_1 \sim \int_{\infty}^{\xi r} e^{-\tau} \frac{g(\frac{\tau}{r} e^{i\alpha})}{\frac{\tau}{r}} \cdot \frac{d\tau}{r} .$$

We now substitute for g its expansion near infinity, i.e.

$$I_1 \sim \int_{\infty}^{\xi r} e^{-\tau} e^{i(m+\frac{1}{2})\alpha} \frac{\tau^{m+\frac{1}{2}}}{r^{m+\frac{1}{2}}} + \frac{1}{r^{m-\frac{1}{2}}} O(\tau^{m-\frac{1}{2}} \ln \tau) \frac{d\tau}{\tau} \sim \frac{c}{r^{m+\frac{1}{2}}} .$$

This singularity, however, is too large to be admitted (since $m \geq 1$). Hence we must put $c_1 = 0$ to remove it.

[3.4] $$I_2 = \int_{-i\infty}^{0} e^{z\zeta} \frac{g_o(\zeta)\tilde{g}_1(\zeta)}{\zeta} d\zeta \quad \text{where}$$

$$\tilde{g}_1(\zeta) = \frac{1}{2\pi i} \int_{0}^{-\infty} \frac{1}{t-\zeta} \left\{ \frac{2i}{\pi} \cdot \frac{t \int_{-\infty}^{0} f(\alpha)[e^{-\alpha t} - 1]d\alpha}{(t+ik)g_o(t)} \right\} dt .$$

To investigate the behavior of I_2 as $|z| \longrightarrow \infty$, we must first obtain the behavior of $\tilde{g}_1(\zeta)$ at $\zeta = 0, \infty$. Hence let

$$h(t) = \frac{t \int_{-\infty}^{0} f(\alpha)[e^{-\alpha t} - 1]d\alpha}{(t+ik)g_o(t)} .$$

Assume that $f(\alpha) \sim \frac{1}{(-\alpha)^{1+\varepsilon}}$, as $\alpha \longrightarrow -\infty$ and f integrable at $\alpha = 0$. Also that $\int_{-\infty}^{0} f(\alpha)d\alpha \neq 0$ for the present.

(1) Let $t \longrightarrow 0^-$.

$$\int_{-\infty}^{0} f(\alpha)[e^{-\alpha t}-1]d\alpha = \int_{-k}^{0} f(\alpha)[e^{-\alpha t}-1]d\alpha + \int_{-\infty}^{-k} f(\alpha)[e^{-\alpha t}-1]d\alpha \quad .$$

For any k, the first term is $O(t)$. Choose the k large enough so that we may replace $f(\alpha)$ by $\frac{1}{(-\alpha)^{1+\varepsilon}}$ in the second integral. Hence

$$\int_{-\infty}^{0} f(\alpha)[e^{-\alpha t}-1]d\alpha \sim O(t) + \int_{-\infty}^{-k} \frac{e^{-\alpha t}-1}{(-\alpha)^{1+\varepsilon}}\, d\alpha \quad .$$

Now, put $\alpha t = \tau$:

$$\int_{-\infty}^{-k} \frac{e^{-\alpha t}-1}{(-\alpha)^{1+\varepsilon}}\, d\alpha = \int_{\infty}^{-kt} \frac{e^{-\tau}-1}{(-\frac{\tau}{t})^{1+\varepsilon}} \cdot \frac{d\tau}{t} = (-t)^{\varepsilon} \int_{\infty}^{-kt} (e^{-\tau}-1)\, \frac{d\tau}{\tau^{1+\varepsilon}}$$

$$= (-t)^{\varepsilon}\, \frac{e^{-\tau}-1}{-\varepsilon\tau^{\varepsilon}}\Bigg|_{\infty}^{-kt} - \frac{1}{\varepsilon} \int_{\infty}^{-kt} e^{-\tau}\, \frac{d\tau}{\tau^{\varepsilon}}$$

$$= (-t)^{\varepsilon}\, \frac{e^{kt}-1}{-\varepsilon(-kt)^{\varepsilon}} - \frac{(-t)^{\varepsilon}}{\varepsilon} \int_{\infty}^{-kt} e^{-\tau}\, \frac{d\tau}{\tau^{\varepsilon}}$$

$$= \frac{e^{kt}-1}{-\varepsilon(-k)^{\varepsilon}} - \frac{(-t)^{\varepsilon}}{\varepsilon} \int_{\infty}^{-kt} e^{-\tau}\, \frac{d\tau}{\tau^{\varepsilon}} \quad .$$

The first term is again $O(t)$. In the second term of $\varepsilon < 1$, we may let $t \longrightarrow 0$ in the integral and get $\int_{\infty}^{0} e^{-\tau}\, \frac{d\tau}{\tau^{\varepsilon}} = \text{constant}$. Hence second term is $O[(-t)^{\varepsilon}]$. Thus for $0 < \varepsilon < 1$,

$$\int_{-\infty}^{0} f(\alpha)[e^{-\alpha t}-1]d\alpha = O[(-t)^{\varepsilon}] \quad .$$

Now if $\varepsilon = 1$, we integrate by parts again, i.e.

$$-t\int_{\infty}^{-kt} e^{-\tau}\frac{d\tau}{\tau} = -t\, e^{-\tau}\ln\tau\Big|_{\infty}^{-kt} + \int_{\infty}^{-kt} e^{-\tau}\ln\tau\, d\tau$$

$$= -te^{kt}\ln(-kt) - t\int_{\infty}^{-kt} e^{-\tau}\ln\tau\, d\tau = O[t\ln(-t)] \text{ as } t\longrightarrow 0^{-}.$$

Finally if $\varepsilon > 1$, repeated integrations by parts show that the lowest order term we get is $O(t)$.

Thus we have the results:

(1) as $t \longrightarrow 0$

$$h(t) \sim \text{(a)}\ \frac{(-t)^{\varepsilon}}{t^{n-2}}, \quad 0 < \varepsilon < 1$$

$$\text{(b)}\ \frac{\ln(-t)}{t^{n-3}}, \quad \varepsilon = 1$$

$$\text{(c)}\ \frac{1}{t^{n-3}}, \quad \varepsilon > 1$$

(2) as $t \longrightarrow -\infty$,

$$h(t) \sim \frac{c}{(-t)^{n-\frac{1}{2}}}$$

where

$$c = -i\int_{-\infty}^{0} f(\alpha)d\alpha \neq 0$$

by assumption since

$$\int_{-\infty}^{0} f(\alpha)[e^{-\alpha t}-1]d\alpha \longrightarrow -\int_{-\infty}^{0} f(\alpha)d\alpha$$

and $g_o(t) \sim t^{n-\frac{1}{2}}$ as $t \longrightarrow -\infty$. Hence from (1) and (2) it is evident that for convergence of the integral defining $\tilde{g}_1(\zeta)$, we must have $1 \leq n \leq 3$.

Now,

$$\tilde{g}_1(\zeta) = \frac{1}{\pi^2} \int_0^{-\infty} \frac{h(t)}{t-\zeta}\, dt$$

where h(t) has the above properties. We must consider cases (a), (b), (c) for each value of n = 1,2,3 for $\zeta \to 0$, $\zeta \to \infty$.

(1) let $\zeta \to 0$.

For n = 1,2, obviously, and cases (a), (b), (c) we have $\tilde{g}_1(\zeta) \to$ constant since the singularity at the origin, if one occurs, is integrable.

For n = 3,

(a) $h(t) \sim \frac{1}{(-t)^{1-\varepsilon}}$ as $t \to 0$; $h(t) \sim \frac{c}{(-t)^{5/2}}$ as $t \to \infty$.

$$\tilde{g}_1(\zeta) = \frac{1}{\pi^2} \int_0^{-\delta} \frac{h(t)}{t-\zeta}\, dt + \int_{-\delta}^{-\infty} \frac{h(t)}{t-\zeta}\, dt \quad .$$

Now for any $\delta > 0$, however small, the second interval is O(1) as $\zeta \to 0$. Let δ be so small that we may replace h(t) by its asymptotic value near t = 0. Thus,

$$\tilde{g}_1(\zeta) \sim \frac{1}{\pi^2} \int_0^{-\delta} \frac{1}{(-t)^{1-\varepsilon}} \cdot \frac{1}{t-\zeta}\, dt + O(1) \quad .$$

Put $t = \zeta\tau$. Therefore

$$\tilde{g}_1(\zeta) \sim \frac{1}{\pi^2} \cdot \frac{1}{\zeta^{1-\varepsilon}} \int_0^{-\delta/\zeta} \frac{1}{(-\tau)^{1-\varepsilon}} \cdot \frac{d\tau}{\tau-1} + O(1) \sim \frac{\text{const.}}{\zeta^{1-\varepsilon}} \text{ , as } \zeta \to 0 \text{ .}$$

(b) $h(t) \sim \ln(-t)$ as $t \to 0^-$; $h(t) \sim \frac{c}{(-t)^{5/2}}$ as $t \to -\infty$.

Similarly to (a),

$$\tilde{g}_1(\zeta) \sim \frac{1}{\pi^2} \int_0^{-\delta} \ln(-t)\, \frac{dt}{t-\zeta} + O(1) \text{ .}$$

Now,

$$\int_1^{\zeta} d\zeta \int_0^{-\delta} \ln(-t)\,\frac{dt}{(t-\zeta)^2} = \int_0^{-\delta} \ln(-t)\,\frac{dt}{t-\zeta} - \int_0^{-\delta} \ln(-t)\,\frac{dt}{t-1} \quad .$$

As $\zeta \longrightarrow 0$,

$$\int_1^{\zeta} d\zeta \int_0^{-\delta} \ln(-t)\,\frac{dt}{(t-\zeta)^2} \sim \int_0^{-\delta} \ln(-t)\,\frac{dt}{t-\zeta} + c \quad .$$

Therefore

$$\tilde{g}_1(\zeta) \sim \frac{1}{\pi^2}\int_1^{\zeta} d\zeta \int_0^{-\delta} \ln(-t)\,\frac{dt}{(t-\zeta)^2} \quad .$$

Put $t = \zeta\tau$,

$$\tilde{g}_1(\zeta) \sim \frac{1}{\pi^2}\int_1^{\zeta} d\zeta \int_0^{-\delta/\zeta} \ln(-\zeta\tau)\,\frac{d\tau}{\zeta(\tau-1)^2}$$

$$= \frac{1}{\pi^2}\int_1^{\zeta} \frac{\ln\zeta}{\zeta}\,d\zeta \int_0^{-\delta/\zeta} \frac{d\tau}{(\tau-1)^2} + \frac{1}{\pi^2}\int_1^{\zeta} \frac{d\zeta}{\zeta} \int_0^{-\delta/\zeta} \frac{\ln(-\tau)d\tau}{(\tau-1)^2} \quad .$$

Therefore

$$\tilde{g}_1(\zeta) \sim c(\ln\zeta)^2, \text{ as } \zeta \longrightarrow 0 \quad .$$

(c) $h(t) \sim 1$ as $t \longrightarrow 0^-$; $h(t) \sim \frac{c}{(-t)^{5/2}}$ as $t \longrightarrow -\infty$.

$$\tilde{g}_1(\zeta) \sim \frac{1}{\pi^2}\int_0^{-\delta} \frac{dt}{t-\zeta} \sim c \ln\zeta, \text{ as } \zeta \longrightarrow 0.$$

(2) Let $\zeta \longrightarrow \infty$.

$$\tilde{g}_1(\zeta) = \frac{1}{\pi^2}\int_0^{-\infty} \frac{h(t)}{t-\zeta} \quad .$$

Put $\frac{1}{\zeta} = \rho$, $\frac{1}{t} = \tau$.

$$\tilde{g}_1(\zeta) = -\frac{1}{\pi^2}\cdot\rho\int_{-\infty}^{0} \frac{\frac{1}{\tau}h(\frac{1}{\tau})d\tau}{\rho-\tau} \quad .$$

As in (1), now, the behavior of $\tilde{g}_1(\zeta)$ as $\zeta \to \infty$ is given by the behavior of the integral as $\rho \to 0$. The behavior of $\frac{1}{\tau} h(\frac{1}{\tau})$ as $\tau \to 0^-$, $\tau \to -\infty$ is:

$$(1) \qquad \frac{1}{\tau} h\left(\frac{1}{\tau}\right) \sim c_1 (-\tau)^{n-\frac{3}{2}}, \quad \text{as } \tau \to 0^-.$$

$$(2) \qquad \frac{1}{\tau} h\left(\frac{1}{\tau}\right) \sim \begin{cases} (a) \quad \dfrac{\tau^{n-3}}{(-\tau)^{\varepsilon}}, & 0 < \varepsilon < 1 \\ (b) \quad -\tau^{n-4} \ln(-\tau), & \varepsilon = 1 \\ (c) \quad \tau^{n-4}, & \varepsilon > 1, \quad \text{as } \tau \to -\infty. \end{cases}$$

Now, for $n = 2,3$ and cases (a), (b), (c) we have $\tilde{g}_1(\zeta) \sim \text{const.}\, \rho$, as $\rho \to 0$, since we may let $\rho \to 0$ under integral sign, the resulting function being integrable.

For $n = 1$, cases (a), (b), (c), we have

$$\tilde{g}_1(\zeta) = -\frac{\rho}{\pi^2} \int_{-\infty}^{-\delta} \frac{\frac{1}{\tau} h\left(\frac{1}{\tau}\right) d\tau}{\rho - \tau} - \frac{\rho}{\pi^2} \int_{-\delta}^{0} \frac{\frac{1}{\tau} h\left(\frac{1}{\tau}\right) d\tau}{\rho - \tau}.$$

Now for any $\delta > 0$, first integral is $O(\rho)$. Hence, as before,

$$\tilde{g}_1(\zeta) \sim O(\rho) - \frac{\rho c_1}{\pi^2} \int_{-\delta}^{0} \frac{1}{(-\tau)^{1/2}} \cdot \frac{1}{\rho - \tau} d\tau = O(\rho) + O(\rho^{1/2}) \quad \text{as } \rho \to 0.$$

Therefore

$$\tilde{g}_1(\zeta) \sim \text{const.}\, \rho^{1/2}, \quad \text{as } \rho \to 0$$

or

$$\tilde{g}_1(\zeta) \sim \frac{\text{const.}}{\zeta^{1/2}}, \qquad \text{as } \zeta \to \infty.$$

Thus we have the results for $\tilde{g}_1(\zeta)$:

For $n = 1,2$, cases (a), (b), (c): $\tilde{g}_1(\zeta) = O(1)$, $\zeta \to 0$.

$n = 3$, case (a): $\tilde{g}_1(\zeta) = O(\frac{1}{\zeta^{1-\varepsilon}})$ as $\zeta \to 0$.

$n = 3$, case (b): $\tilde{g}_1(\zeta) = O(\ln^2\zeta)$ as $\zeta \to 0$.

$n = 3$, case (c): $\tilde{g}_1(\zeta) = O(\ln \zeta)$ as $\zeta \to 0$.

For $n = 2,3$, cases (a), (b), (c): $\tilde{g}_1(\zeta) = O(\frac{1}{\zeta})$ as $\zeta \to \infty$.

$n = 1$, cases (a), (b), (c): $\tilde{g}_1(\zeta) = O(\frac{1}{\zeta^{1/2}})$ as $\zeta \to \infty$.

We now have enough information to consider the integral

$$I_2 = \int_{-i\infty}^{0} e^{z\zeta} \frac{g_o(\zeta)\tilde{g}_1(\zeta)}{\zeta} d\zeta$$

where

$$g_o(\zeta) = \zeta^{n-\frac{1}{2}} \psi(\zeta) .$$

First we observe that $n \neq 1$, since then

$$\frac{g_o(\zeta)\tilde{g}_1(\zeta)}{\zeta} = O(\frac{1}{\zeta}) , \quad \zeta \to 0$$

for cases (a), (b), (c), and hence is not integrable at $\zeta = 0$. Next we see that $n \neq 3$, for then

$$\frac{g_o(\zeta)\tilde{g}_1(\zeta)}{\zeta} = O(\zeta^{1/2}) \quad \text{as} \quad \zeta \to \infty$$

and I_2 will have a singularity as $z \to 0$ of order $z^{-3/2}$, which is too big to be admissible. Thus $n = 2$. For this n and any case (a), (b), (c):

$$f(\zeta) = \frac{g_o(\zeta)\tilde{g}_1(\zeta)}{\zeta} = \begin{cases} O(1) & \text{as } \zeta \to 0 \\ O(\frac{1}{\zeta^{1/2}}) & \text{as } \zeta \to \infty . \end{cases}$$

By shifting path to get a negative real exponential, as before, we get:

$$I_2 = O(\frac{1}{z}) \qquad \text{as } z \longrightarrow \infty$$

$$I_2 = O(\frac{1}{z^{1/2}}) \quad \text{as } z \longrightarrow 0 \ .$$

[3.5] In the preceding, we assumed that

$$\int_{-\infty}^{0} f(\alpha)d\alpha \neq 0 \ .$$

This assumption will now be removed. Now assume

$$\int_{-\infty}^{0} f(\alpha)d\alpha = 0 \ .$$

This condition only affects the behavior of $h(t)$ at $t = -\infty$.

For now,

$$h(t) = \frac{t\int_{-\infty}^{0} f(\alpha)e^{-\alpha t}d\alpha}{(t+ik)g_o(t)} \ .$$

The behavior of the integral as $t \longrightarrow -\infty$ depends upon the behavior of $f(\alpha)$ near $\alpha = 0$. Assume, since $f(\alpha)$ must remain integrable, that $f(\alpha) \sim \frac{c_1}{\alpha^{\delta}} + \ldots$ near $\alpha = 0$, where $0 \leq \delta \leq 1$.

In integral, put $\alpha t = \tau$, and consider range $(0,-\xi)$ as before. Thus

$$\int_{-\xi}^{0} f(\alpha)e^{-\alpha t}d\alpha = \int_{-\xi t}^{0} f(\frac{\tau}{t})e^{-\tau}\frac{d\tau}{t} \sim c_1 \int_{-\xi t}^{0} (\frac{t}{\tau})^{\delta} e^{-\tau}\frac{d\tau}{t} \frac{c_1}{t^{1-\delta}} \ .$$

Hence, as $t \longrightarrow -\infty$,

$$h(t) \sim \frac{c_1 i}{(-t)^{n+1/2-\delta}} \ .$$

As $t \longrightarrow 0^-$, we have as before:

$$h(t) \sim \begin{cases} \text{(a)} \quad \dfrac{(-t)^{\varepsilon}}{t^{n-2}} , & 0 < \varepsilon < 1 . \\ \text{(b)} \quad \dfrac{\ln(-t)}{t^{n-3}} , & \varepsilon = 1 . \\ \text{(c)} \quad \dfrac{1}{t^{n-3}} , & \varepsilon > 1 . \end{cases}$$

Now if $\delta < \frac{1}{2}$, we may admit $n = 0$ in addition to $1,2,3$ as values. If $\delta \geq \frac{1}{2}$, we have only cases $n = 1,2,3$. We now consider

$$\tilde{g}_1(\zeta) = \frac{1}{\pi^2} \int_0^{-\infty} \frac{h(t)}{t - \zeta} \, dt .$$

(A) Let $\zeta \longrightarrow 0$.

(1) $n = 0,1,2$; cases (a), (b), (c)

$$\tilde{g}_1(\zeta) = O(1)$$

(2) $n = 3$; case (a)

$$\tilde{g}_1(\zeta) = O\left(\frac{1}{\zeta^{1-\varepsilon}}\right)$$

(3) $n = 3$; case (b)

$$\tilde{g}_1(\zeta) = O(\ln^2 \zeta)$$

(4) $n = 3$; case (c)

$$\tilde{g}_1(\zeta) = O(\ln \zeta)$$

(B) Let $\zeta \longrightarrow \infty$.

$$\tilde{g}_1(\zeta) = -\frac{\rho}{\pi^2} \int_{-\infty}^{0} \frac{\frac{1}{\tau} h(\frac{1}{\tau}) d\tau}{\rho - \tau} , \quad \rho = \frac{1}{\zeta} .$$

Now behavior of $\frac{1}{\tau} h(\frac{1}{\tau})$ is given by following:

$$\frac{1}{\tau} h(\frac{1}{\tau}) \sim -c_1 i(-\tau)^{n-1/2-\delta} , \quad \text{as } \tau \longrightarrow 0^- .$$

$$\frac{1}{\tau} h\left(\frac{1}{\tau}\right) \sim \begin{cases} \text{(a)} & \dfrac{\tau^{n-3}}{(-\tau)^{\varepsilon}}, & 0 < \varepsilon < 1 \\ \text{(b)} & -\tau^{n-4} \ln(-\tau), & \varepsilon = 1 \\ \text{(c)} & \tau^{n-4}, & \varepsilon > 1, \end{cases} \quad \text{as } \tau \to -\infty .$$

(1) $n = 2,3$; cases (a), (b), (c).

$$\tilde{g}_1(\zeta) = O\left(\frac{1}{\zeta}\right) .$$

(2) $n = 1$; cases (a), (b), (c); $\delta < \frac{1}{2}$.

$$\tilde{g}_1(\zeta) = O\left(\frac{1}{\zeta}\right) .$$

(3) $n = 1$; cases (a), (b), (c); $\delta \geq \frac{1}{2}$.

$$\tilde{g}_1(\zeta) = O\left(\frac{1}{\zeta^{3/2-\delta}}\right) .$$

(4) $n = 0$; cases (a), (b), (c); $\delta < \frac{1}{2}$.

$$\tilde{g}_1(\zeta) = O\left(\frac{1}{\zeta^{1/2-\delta}}\right) .$$

We now consider again:

$$I_2 = \int_{-i\infty}^{0} e^{z\zeta} \frac{g_o(\zeta)\tilde{g}_1(\zeta)}{\zeta} d\zeta \quad \text{where} \quad g_o(\zeta) = \zeta^{n-1/2}\psi(\zeta) .$$

Now $n \neq 0,1$ for then

$$\frac{g_o(\zeta)\tilde{g}_1(\zeta)}{\zeta} \sim \frac{1}{\zeta^2}, \frac{1}{\zeta}, \quad \zeta \to 0 ,$$

respectively, which is too large a singularity at $\zeta = 0$. Also, $n \neq 3$, for then

$$\frac{g_o(\zeta)\tilde{g}_1(\zeta)}{\zeta} \sim \zeta^{1/2} \quad \text{as} \quad \zeta \to \infty$$

which leads, as before, to a singularity at z = 0 which is too large. Thus n = 2. We get then the same results as before, viz:

$$I_2 = O(\frac{1}{z}) , \qquad z \longrightarrow \infty$$

$$I_2 = O(\frac{1}{z^{1/2}}) , \quad z \longrightarrow 0 .$$

[3.6] Finally, we consider

$$I_3 = \int_{-\infty}^{0} f(\alpha) \log \frac{(x-\alpha)^2+y^2}{x^2+y^2} d\alpha .$$

(A) As $x,y \longrightarrow 0$.

Evidently,

$$I_3 \sim 2 \int_{-\infty}^{0} f(\alpha) \log (-\alpha) d\alpha - \int_{-\infty}^{0} f(\alpha) d\alpha \cdot \log (x^2+y^2)$$

since the first integral exists

$$\left\{ f(\alpha) \sim \frac{1}{(-\alpha)^{1+\varepsilon}} , \quad \alpha \longrightarrow -\infty \right\} .$$

Thus I_3 has a logarithmic singularity if

$$\int_{-\infty}^{0} f(\alpha) d\alpha \neq 0 .$$

This might be explained as follows:

$$\psi = -\mathrm{Re}\ i\ I_2 - \frac{1}{2\pi} I_3$$

$$\psi \sim \frac{1}{2\pi} \int_{-\infty}^{0} f(\alpha) d\alpha \cdot \log (x^2+y^2) + \ldots .$$

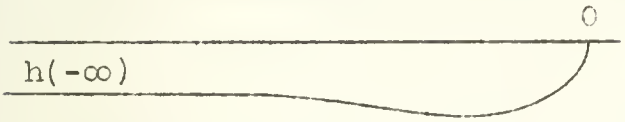
Now since $f(\alpha) = -s_o h_\alpha$,

$$\int_{-\infty}^{0} f(\alpha)d\alpha = -s_o \int_{-\infty}^{0} h_\alpha d\alpha = -s_o[h(0) - h(-\infty)] \quad .$$

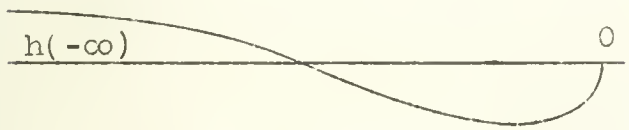
Assume $h(0) = 0$. Thus

$$\int_{-\infty}^{0} f(\alpha)d\alpha = s_o h(-\infty) \quad .$$

Now if $h(-\infty) \neq 0$, the shape of the hull looks like



if $h(-\infty) < 0$, and like



if $h(-\infty) > 0$. The singularity is thus a sink or a source respectively, which takes away or adds exactly enough fluid per unit time to compensate for the shape of the hull at $-\infty$. Of course, the second case could not occur for then all but a finite part of the ship would be out of the water. Such a condition would lead to a different problem at the outset.

(B) As $x^2 + y^2 \longrightarrow \infty$.

Now,

$$I_3 = \int_{-\infty}^{-k} + \int_{-k}^{0} \quad .$$

For any fixed k, second integral $\longrightarrow 0$ as $x^2+y^2 \longrightarrow \infty$. In the first integral:

(1) Let $x > 0$, $x^2+y^2 > M^2 > 1$.

$$\log \frac{(x-\alpha)^2+y^2}{x^2+y^2} = \log 1 + \frac{\alpha^2}{x^2+y^2} - \frac{2\alpha x}{x^2+y^2}$$

$$< \log 1 + \frac{\alpha^2}{M^2} + \frac{2|\alpha|}{M} < \log (1+|\alpha|)^2 \quad .$$

(2) Let $x = 0$, $x^2+y^2 > M^2 > 1$.

$$\log \frac{(x-\alpha)^2+y^2}{x^2+y^2} = \log \frac{\alpha^2+y^2}{y^2} = \log 1 + \frac{\alpha^2}{y^2}$$

$$< \log (1+\alpha^2) < \log (1+|\alpha|)^2 .$$

(3) Let $-\frac{k}{2} \leq x < 0$, $x^2+y^2 > M^2 > 1$.

$$\log \frac{(x-\alpha)^2+y^2}{x^2+y^2} = \log 1 + \frac{\alpha^2}{x^2+y^2} - \frac{2\alpha x}{x^2+y^2}$$

$$< \log (1+\alpha^2) < \log (1+|\alpha|^2) .$$

Thus for $x \geq -\frac{k}{2}$, $x^2+y^2 > M^2 > 1$,

$$0 \leq \log \frac{(x-\alpha)^2+y^2}{x^2+y^2} < \log (1+|\alpha|)^2 .$$

Therefore

$$\int_{-\infty}^{-k} f(\alpha) \log \frac{(x-\alpha)^2+y^2}{x^2+y^2} \, d\alpha \leq \int_{-\infty}^{-k} |f(\alpha)| \log (1+|\alpha|)^2 d\alpha .$$

(4) Let $x < -\frac{k}{2}$, $y \neq 0$, $x^2+y^2 > M^2 > 1$.

$$\int_{-\infty}^{-k} = \int_{-\infty}^{2x} + \int_{2x}^{-k} .$$

In first part,

$$0 < \log \frac{(x-\alpha)^2+y^2}{x^2+y^2} = \log 1 + \frac{\alpha^2}{x^2+y^2} - \frac{2\alpha x}{x^2+y^2} < \log (1+|\alpha|)^2 .$$

Therefore

$$\int_{-\infty}^{2x} f(\alpha) \log \frac{(x-\alpha)^2+y^2}{x^2+y^2}\, d\alpha < \int_{-\infty}^{2x} |f(\alpha)| \log (1+|\alpha|)^2 d\alpha$$

$$< \int_{-\infty}^{-k} |f(\alpha)| \log (1+|\alpha|)^2 d\alpha \quad .$$

Also,

$$\int_{2x}^{-k} f(\alpha) \log \frac{(x-\alpha)^2+y^2}{x^2+y^2}\, d\alpha < -\log \frac{y^2}{x^2+y^2} \int_{2x}^{-k} |f(\alpha)| d\alpha$$

$$< -\log \frac{y^2}{x^2+y^2} \int_{-\infty}^{-k} |f(\alpha)| d\alpha \quad .$$

If $\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} -\pi < \theta < 0$

$$-\log \frac{y^2}{x^2+y^2} = -\log \sin^2\theta$$

which is finite as long as $\theta > -\pi + \theta_1$, $\theta_1 > 0$.

(5) Let $x < -\frac{k}{2}$, $y = 0$, $x^2+y^2 > M^2 > 1$.

$$\int_{-\infty}^{-k} f(\alpha) \log \left(\frac{x-\alpha}{x}\right)^2 d\alpha = \int_{-\infty}^{2x} + \int_{2x}^{-k} \quad .$$

In first part,

$$0 < \log \left(\frac{x-\alpha}{x}\right)^2 < \log (1+|\alpha|)^2 \quad .$$

Therefore

$$\int_{-\infty}^{2x} f(\alpha) \log \left(\frac{x-\alpha}{x}\right)^2 d\alpha < \int_{-\infty}^{-k} |f(\alpha)| \log (1+|\alpha|)^2 d\alpha \quad .$$

In second part,

$$\int_{2x}^{-k} f(\alpha) \log \left(\frac{x-\alpha}{x}\right)^2 d\alpha$$

$$= \int_{2x}^{-k} (-\alpha)^{1+\frac{\varepsilon}{2}} \cdot f(\alpha) \cdot \frac{1}{(-\alpha)^{1+\frac{\varepsilon}{2}}} \cdot \log \left(\frac{x-\alpha}{x}\right)^2 d\alpha \quad \text{where } 1 > \varepsilon > 0$$

$$\leq \max_{2x \leq \alpha \leq -k} \cdot (-\alpha)^{1+\frac{\varepsilon}{2}} \cdot f(\alpha) \cdot \int_{2x}^{-k} \frac{\left|\log \left(\frac{x-\alpha}{x}\right)^2\right|}{(-\alpha)^{1+\varepsilon/2}} d\alpha \quad .$$

In integral, let $\alpha = \tau x$.

$$\int_{2x}^{-k} \frac{\left|\log \left(\frac{x-\alpha}{x}\right)^2\right| d\alpha}{(-\alpha)^{1+\varepsilon/2}} = \frac{1}{(-x)^{\varepsilon/2}} \int_{2}^{-k/x} \frac{\left|\log (1-\tau)^2\right| d\tau}{\tau^{1+\varepsilon/2}}$$

$$< \frac{1}{(-x)^{\varepsilon/2}} \int_{2}^{0} \frac{\left|\log (1-\tau)^2\right| d\tau}{\tau^{1+\varepsilon/2}} < \frac{A}{(-x)^{\varepsilon/2}} \quad \text{where}$$

$$A = \int_{2}^{0} \frac{\left|\log (1-\tau)^2\right| d\tau}{\tau^{1+\varepsilon/2}} \quad .$$

Hence

$$\int_{2x}^{-k} f(\alpha) \log \left(\frac{x-\alpha}{x}\right)^2 d\alpha \leq \frac{A}{(-x)^{\varepsilon/2}} \cdot \max_{2x \leq \alpha \leq -k} (-\alpha)^{1+\varepsilon/2} f(\alpha) \quad .$$

From (1)-(5) above, we see that by first choosing the k large enough we may make the integral

$$\int_{-\infty}^{-k} f(\alpha) \log \frac{(x-\alpha)^2+y^2}{x^2+y^2} d\alpha$$

as small as we please, even uniformly small as long as $\theta > -\pi + \theta_1$ in (4) above, no matter what x or y are, as long as

$x^2+y^2 > 1$. Thus as $x^2+y^2 \longrightarrow \infty$, we will have $I_3 \longrightarrow 0$ uniformly in any sector excluding the negative axis. Also, even on negative axis, $I_3 \longrightarrow 0$.

[3.7] Thus, finally, for the solution to the problem

$$\psi = -\mathrm{Re}\ i\ I_2 - \frac{1}{2\pi} I_3 \quad ,$$

we will have:

(1) $\psi \longrightarrow 0$ as $x^2+y^2 \longrightarrow \infty$ along any ray in lower half-plane.

(2) $\psi \sim \dfrac{s_o h(-\infty)}{2\pi} \log (x^2+y^2) + \mathrm{Re}\ \dfrac{\text{const.}}{\sqrt{z}}$, as $x^2+y^2 \longrightarrow 0$, $z = x+iy$, if $h(-\infty) \neq 0$,

$$\psi \sim \mathrm{Re}\ \frac{\text{const.}}{\sqrt{z}} \quad \text{if} \quad h(-\infty) = 0 \quad .$$

We now append proofs of certain facts which were omitted in the preceding.

[4.1] $$H(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{1}{t-\zeta} \ln \frac{t+i}{t-i}\, dt \quad .$$

Let ζ be real and negative.

$$H(\zeta) = -\frac{1}{2} \ln \frac{\zeta+i}{\zeta-i} + \mathrm{P.V.} \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{1}{t-\zeta} \ln \frac{t+i}{t-i}\, dt$$

$$\mathrm{P.V.} \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{1}{t-\zeta} \ln \frac{t+i}{t-i}\, dt = \lim_{\varepsilon \longrightarrow 0} \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\zeta-\varepsilon} + \frac{1}{2\pi i} \int_{\zeta+\varepsilon}^{0} \right\}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\zeta-\varepsilon}\frac{1}{t-\zeta}\ln\frac{t+i}{t-i}\,dt=\frac{1}{2\pi i}\int_{-\infty}^{\zeta-\varepsilon}\ln\frac{t+i}{t-i}\,d\ln(\zeta-t)$$

$$=\frac{1}{2\pi i}\left\{\ln\frac{t+i}{t-i}\ln(\zeta-t)\Big|_{-\infty}^{\zeta-\varepsilon}+2i\int_{-\infty}^{\zeta-\varepsilon}\ln(\zeta-t)\frac{dt}{t^2+1}\right\}$$

$$=\frac{1}{2\pi i}\left\{\ln\frac{(\zeta-\varepsilon)+i}{(\zeta-\varepsilon)-i}\ln\varepsilon+2i\int_{-\infty}^{\zeta-\varepsilon}\ln(\zeta-t)\frac{dt}{t^2+1}\right\}$$

$$\frac{1}{2\pi i}\int_{\zeta+\varepsilon}^{0}\frac{1}{t-\zeta}\ln\frac{t+i}{t-i}\,dt=\frac{1}{2\pi i}\int_{\zeta+\varepsilon}^{0}\ln\frac{t+i}{t-i}\,d\ln(t-\zeta)$$

$$=\frac{1}{2\pi i}\left\{\ln(-1)\ln(-\zeta)-\ln\frac{(\zeta+\varepsilon)+i}{(\zeta+\varepsilon)-i}\ln\varepsilon+2i\int_{\zeta+\varepsilon}^{0}\ln(t-\zeta)\frac{dt}{t^2+1}\right\}.$$

Therefore

$$\text{P.V.}\ \frac{1}{2\pi i}\int_{-\infty}^{0}\frac{1}{t-\zeta}\ln\frac{t+i}{t-i}\,dt=\frac{\ln(-1)}{2\pi i}\ln(-\zeta)$$

$$+\frac{1}{\pi}\int_{-\infty}^{\zeta}\ln(\zeta-t)\frac{dt}{t^2+1}+\frac{1}{\pi}\int_{\zeta}^{0}\ln(t-\zeta)\frac{dt}{t^2+1}\ .$$

Let $\zeta\tau = t$ in the integrals

$$\text{P.V.}\ (\zeta)=\frac{\ln(-1)}{2\pi i}\ln(-\zeta)$$

$$+\frac{\zeta\ln(-\zeta)}{\pi}\int_{\infty}^{1}\frac{d\tau}{\zeta^2\tau^2+1}+\frac{\zeta\ln(-\zeta)}{\pi}\int_{1}^{0}\frac{d\tau}{\zeta^2\tau^2+1}$$

$$+\frac{\zeta}{\pi}\int_{\infty}^{1}\ln(\tau-1)\frac{d\tau}{\zeta^2\tau^2+1}+\frac{\zeta}{\pi}\int_{1}^{0}\ln(1-\tau)\frac{d\tau}{\zeta^2\tau^2+1}$$

$$= \frac{\ln(-1)\ln(-\zeta)}{2\pi i} + \frac{\ln(-\zeta)}{2} + \frac{\zeta}{\pi}\int_{\infty}^{1} \ln(\tau-1)\,\frac{d\tau}{\zeta^2\tau^2+1}$$

$$+ \frac{\zeta}{\pi}\int_{1}^{0} \ln(1-\tau)\,\frac{d\tau}{\zeta^2\tau^2+1}$$

$$= \frac{\zeta}{\pi}\int_{\infty}^{1} \ln(\tau-1)\,\frac{d\tau}{\zeta^2\tau^2+1} + \frac{\zeta}{\pi}\int_{1}^{0} \ln(1-\tau)\,\frac{d\tau}{\zeta^2\tau^2+1}\ ,$$

since $\ln(-1) = -\pi i$.

(a) Let $\zeta \to 0^-$. Second integral is obviously $O(\zeta)$. In first, put $\zeta\tau = t$. We get the integrals,

$$\frac{1}{\pi}\int_{-\infty}^{\zeta} \frac{\ln(\zeta-t)}{t^2+1}\,dt - \frac{\ln(-\zeta)}{\pi}\int_{-\infty}^{\zeta} \frac{dt}{t^2+1} \sim -\frac{1}{2}\ln(-\zeta) + O[\ln(-\zeta)] \ .$$

Therefore P.V. $(\zeta) \sim -\frac{1}{2}\ln(-\zeta)$ as $\zeta \to 0^-$ along negative axis.

(b) Let $\zeta \to -\infty$.

$$\frac{\zeta}{\pi}\int_{\infty}^{1} \ln(\tau-1)\,\frac{d\tau}{\zeta^2\tau^2+1} = \frac{1}{\pi\zeta}\int_{\infty}^{1} \ln(\tau-1)\,\frac{d\tau}{\tau^2+\frac{1}{\zeta^2}} \sim \frac{c}{\zeta} \quad \text{as } \zeta \to -\infty \ .$$

$$\frac{\zeta}{\pi}\int_{1}^{0} \ln(1-\tau)\,\frac{d\tau}{\zeta^2\tau^2+1} = \frac{\rho}{\pi}\int_{1}^{0} \ln(1-\tau)\,\frac{d\tau}{\tau^2+\rho^2} \quad \text{by putting } \frac{1}{\zeta} = \rho \ .$$

Now,

$$\frac{\rho}{\pi}\int_{1}^{0} \ln(1-\tau)\,\frac{d\tau}{\tau^2+\rho^2} = \frac{\rho}{\pi}\int_{1}^{1/2} \ln(1-\tau)\,\frac{d\tau}{\tau^2+\rho^2} + \frac{\rho}{\pi}\int_{1/2}^{0} \ln(1-\tau)\,\frac{d\tau}{\tau^2+\rho^2}$$

$$\leq \frac{-4\rho}{\pi}\int_{1}^{1/2} -\ln(1-\tau)\,d\tau - \frac{\rho}{\pi}\int_{1/2}^{0} \frac{d\tau}{\tau^2+\rho^2}$$

$$\leq (-\rho)\ln(-\rho) = \frac{-1}{\zeta}\ln(-\zeta) \quad \text{as } \zeta \to -\infty \ .$$

Thus $H(\zeta)$, even for ζ on the negative axis, behaves the same as for other ζ. Thus the shift of path to the negative axis in [2.6] is valid for it gives a convergent integral.

[4.2] $$\int_{-\infty}^{0} f(\alpha)[e^{-\alpha t} - 1]d\alpha \ , \quad t \longrightarrow 0^{-}$$

$$f(\alpha) = \frac{1}{(-\alpha)^{1+\varepsilon}} + O\left[\frac{1}{(-\alpha)^{1+\varepsilon}}\right] \ , \quad \varepsilon > 0 \ , \quad \text{as } \alpha \longrightarrow -\infty \ .$$

(1) Let $\varepsilon < 1$.

$$\int_{-\infty}^{0} f(\alpha)[e^{-\alpha t} - 1]d\alpha = \int_{-\infty}^{-A} + \int_{-A}^{0} \ , \quad A > 0$$

$$\left| \int_{-A}^{0} f(\alpha)[e^{-\alpha t} - 1]d\alpha \right| = \left| \int_{-A}^{0} f(\alpha)\left[\alpha t - \frac{\alpha^2 t^2}{2} + \ldots\right]d\alpha \right|$$

$$\leq K\left[At + \frac{A^2 t^2}{2} + \ldots\right]$$

$$\text{where } K = \int_{-\infty}^{0} |f(\alpha)|\,d\alpha$$

$$\int_{-\infty}^{-A} f(\alpha)[e^{-\alpha t} - 1]d\alpha = \int_{-\infty}^{-A} [e^{-\alpha t} - 1]\,\frac{d\alpha}{(-\alpha)^{1+\varepsilon}} + R$$

$$\text{where } |R| \leq |e^{At} - 1| \cdot \left(\frac{1}{A^{\varepsilon}}\right)$$

$$\int_{-\infty}^{-A} [e^{-\alpha t} - 1] \frac{d\alpha}{(-\alpha)^{1+\varepsilon}} = (-t)^{\varepsilon} \int_{-At}^{\infty} [e^{-\tau} - 1] \frac{d\tau}{\tau^{1+\varepsilon}}$$

$$= (-t)^{\varepsilon} \left[\frac{e^{-\tau} - 1}{-\varepsilon \tau^{\varepsilon}} \Big|_{-At}^{\infty} - \frac{1}{\varepsilon} \int_{-At}^{\infty} \frac{e^{-\tau}}{\tau^{\varepsilon}} d\tau \right]$$

$$= \frac{e^{-At} - 1}{\varepsilon A^{\varepsilon}} - \frac{(-t)^{\varepsilon}}{\varepsilon} \int_{-At}^{\infty} \frac{e^{-\tau}}{\tau^{\varepsilon}} d\tau \quad .$$

If we let $A = -\log t$, we see that the lowest order term is $c(-t)^{\varepsilon}$ as $t \longrightarrow 0^{-}$.

Thus,

$$\int_{-\infty}^{0} f(\alpha)[e^{-\alpha t} - 1]d\alpha \sim c(-t)^{\varepsilon} + O[(-t)^{2}] \quad .$$

(2) Let $\varepsilon = 1$.

$$\int_{-\infty}^{0} f(\alpha)[e^{-\alpha t} - 1]d\alpha = \int_{-\infty}^{-A} + \int_{-A}^{0} \quad , \quad A > 0 \, .$$

Again

$$\left| \int_{-A}^{0} f(\alpha)[e^{-\alpha t} - 1]d\alpha \right| \leq K\left[At + \frac{A^{2}t^{2}}{2} + \ldots\right]$$

and

$$\int_{-\infty}^{-A} f(\alpha)[e^{-\alpha t} - 1]d\alpha = \int_{-\infty}^{-A} [e^{-\alpha t} - 1] \frac{d\alpha}{(-\alpha)^{2}} + R$$

$$\text{where } |R| \leq |e^{-At} - 1| \cdot \frac{1}{A} \quad .$$

Now,

$$\int_{-\infty}^{-A} [e^{-\alpha t} - 1] \frac{d\alpha}{(-\alpha)^2} = (-t) \int_{-At}^{\infty} [e^{-\tau} - 1] \frac{d\tau}{\tau^2}$$

$$= (-t) \left[\frac{e^{-At} - 1}{A(t)} + e^{-At} \ln A(-t) - \int_{-At}^{\infty} e^{-\tau} \ln \tau \, d\tau \right]$$

$$= \frac{e^{-At} - 1}{A} - te^{-At} \ln (-t) - te^{-At} \ln A + t \int_{-At}^{\infty} e^{-\tau} \ln \tau \, d\tau \ .$$

If we let $A = \sqrt{-\ln (-t)}$, we see that the lowest order term is $ct \ln (-t)$ as $t \longrightarrow 0^-$.

Thus

$$\int_{-\infty}^{0} f(\alpha)[e^{-\alpha t} - 1]d\alpha \sim ct \ln (-t) + O[t \ln (-t)] \quad \text{as } t \longrightarrow 0^- \ .$$

(3) A similar proof may be obtained when $\varepsilon > 1$.

[4.3]

$$I = \int_{-i\infty}^{0} e^{z\zeta} g(\zeta) d\zeta \ , \quad z \longrightarrow 0 \ , \quad z \longrightarrow \infty \text{ along rays in}$$

lower half-plane where

$$g(\zeta) = \zeta^m + O(\zeta^m) \quad \text{as} \quad \zeta \longrightarrow \infty \ , \quad m > -1$$

$$g(\zeta) = \zeta^n + O(\zeta^n) \quad \text{as} \quad \zeta \longrightarrow 0 \ , \quad n > -1 \ .$$

Shift the path of integration so that the integral becomes:

$$I = \int_{\infty}^{0} e^{-r\rho} g(\rho e^{i\alpha}) e^{i\alpha} d\rho \ .$$

We now investigate this integral as $r \longrightarrow 0, \infty$.

Put $r\rho = \tau$

$$I = \int_{\infty}^{0} e^{-\tau} g(\frac{\tau}{r} e^{i\alpha}) e^{i\alpha} \frac{d\tau}{r} = \frac{e^{i\alpha}}{r} \int_{\infty}^{0} e^{-\tau} g(\frac{\tau}{r} e^{i\alpha}) d\tau .$$

(1) Let $r \longrightarrow \infty$

$$\frac{e^{i\alpha}}{r} \int_{\infty}^{0} e^{-\tau} g(\frac{\tau}{r} e^{i\alpha}) d\tau = \frac{e^{i\alpha}}{r} \left\{ \int_{\infty}^{r} + \int_{r}^{\sqrt{r}} + \int_{\sqrt{r}}^{0} \right\}$$

$$\int_{\sqrt{r}}^{0} e^{-\tau} g(\frac{\tau}{r} e^{i\alpha}) d\tau = \int_{\sqrt{r}}^{0} e^{-\tau} \left\{ \frac{\tau^n}{r^n} \cdot e^{in\alpha} \right\} d\tau + R$$

where

$$|R| < \frac{\varepsilon}{r^n} \int_{\sqrt{r}}^{0} e^{-\tau} \tau^n e^{in\alpha} d\tau < \frac{\varepsilon}{r^n} K , \quad \varepsilon \longrightarrow 0 \quad \text{as } r \longrightarrow \infty .$$

$$\frac{e^{in\alpha}}{r^n} \int_{\sqrt{r}}^{0} e^{-\tau} \tau^n d\tau = \frac{c}{r^n} + O(\frac{1}{r^n})$$

$$\left| \int_{r}^{\sqrt{r}} e^{-\tau} g(\frac{\tau}{r} e^{i\alpha}) d\tau \right| \leq e^{-\sqrt{r}} \int_{r}^{\sqrt{r}} |g(\frac{\tau}{r} e^{i\alpha})| d\tau < Ke^{-\sqrt{r}} << \frac{c}{r^n}$$

$$\int_{\infty}^{r} e^{-\tau} g(\frac{\tau}{r} e^{i\alpha}) d\tau = r \int_{\infty}^{1} e^{-rs} g(se^{i\alpha}) ds$$

$$\left| r \int_{\infty}^{1} e^{-rs} g(se^{i\alpha}) ds \right| \leq re^{-\frac{r}{2}} \int_{\infty}^{1} e^{-r(s-1/2)} |g(se^{i\alpha})| ds$$

$$< re^{-\frac{r}{2}} \int_{\infty}^{1} e^{-(s-1/2)} |g(se^{i\alpha})| ds < K \cdot re^{-\frac{r}{2}} << \frac{1}{r^n} .$$

Hence,

$$I = \frac{c}{r^{n+1}} + O(\frac{1}{r^{n+1}}) \quad \text{as} \quad r \longrightarrow \infty .$$

(2) Let $r \longrightarrow 0$

$$\frac{e^{i\alpha}}{r}\int_{\infty}^{0} e^{-\tau} g\left(\frac{\tau}{r}\, e^{i\alpha}\right) d\tau = \frac{e^{i\alpha}}{r}\left\{\int_{\infty}^{\sqrt{r}} + \int_{\sqrt{r}}^{r} + \int_{r}^{0}\right\}$$

$$\int_{\infty}^{\sqrt{r}} e^{-\tau} g\left(\frac{\tau}{r}\, e^{i\alpha}\right) d\tau = \int_{\infty}^{\sqrt{r}} e^{-\tau}\left\{\frac{\tau^m}{r^m}\, e^{im\alpha}\right\} d\tau + R$$

where

$$|R| < \frac{\varepsilon}{r^m}\int_{\infty}^{\sqrt{r}} e^{-\tau}\, \tau^m\, e^{im\alpha} d\tau < \frac{\varepsilon}{r^m}\,, \quad \varepsilon \longrightarrow 0 \quad \text{as} \quad r \longrightarrow 0$$

$$\int_{\sqrt{r}}^{r} e^{-\tau} g\left(\frac{\tau}{r}\, e^{i\alpha}\right) d\tau \leq \int_{\sqrt{r}}^{r} e^{-\tau}\, \frac{\tau^m}{r^m}\, d\tau < \frac{K}{r^m}\int_{\sqrt{r}}^{r} \tau^m d\tau << \frac{K}{r^m}$$

$$\int_{r}^{0} e^{-\tau} g\left(\frac{\tau}{r}\, e^{i\alpha}\right) d\tau = O(r) << \frac{K}{r^m}\,.$$

Hence

$$I = \frac{c}{r^{m+1}} + O\left(\frac{1}{r^{m+1}}\right) \quad \text{as} \quad r \longrightarrow 0$$

[4.4]

$$I = \int_{0}^{\infty} \frac{f(t)}{t - \zeta}\, dt\,,$$

$\zeta \longrightarrow 0$ along a ray in the lower half-plane. Where

(1) $f(t) \sim \frac{1}{t^{\delta}}$, $t \longrightarrow 0$, $0 < \delta < 1$

(2) $f(t) \sim 1$, $t \longrightarrow 0$

(3) $f(t) \sim \ln t$, $t \longrightarrow 0$

and $f(t) \sim \frac{1}{t^m}$, $m > 0$, $t \longrightarrow \infty$.

(1) $f(t) = \frac{1}{t^\delta} + O(\frac{1}{t^\delta})$, $t \to 0$.

$$I = \int_0^\infty \frac{dt}{(t-\zeta)t^\delta} = \int_0^\infty \frac{g(t)}{t-\zeta}\,dt \text{ where } g(t) = f(t) - \frac{1}{t^\delta} = O(\frac{1}{t^\delta}),\ t \to 0.$$

Thus the highest order term, as $\zeta \to 0$, if I is given by considering

$$\int_0^\infty \frac{dt}{(t-\zeta)t^\delta} = \int_0^A \frac{dt}{(t-\zeta)t^\delta} + \int_A^\infty \frac{dt}{(t-\zeta)t^\delta}$$

$$\int_A^\infty \frac{dt}{(t-\zeta)t^\delta} = \int_A^\infty \left\{1 + \frac{\zeta}{t} + \frac{\zeta^2}{t^2} + \cdots\right\} \frac{dt}{t^{1+\delta}} = \frac{c_1}{A^\delta} + \frac{c_2\zeta}{A^{1+\delta}} + \cdots$$

$$\int_0^A \frac{dt}{(t-\zeta)t^\delta} = \frac{1}{\zeta^\delta}\int_0^{A/\zeta} \frac{d\tau}{(\tau-1)\tau^\delta}\ .$$

Now if we set $A = |\zeta|^{1/2}$ we get that the highest order infinity as $\zeta \to 0$ is $\frac{c}{\zeta^\delta}$. Hence

$$I = \frac{c}{\zeta^\delta} + O(\frac{1}{\zeta^\delta}) \quad \text{as} \quad \zeta \to 0\ .$$

(2) $f(t) = 1 + O(1)$, $t \to 0$. As in (1), we consider

$$I \sim \int_0^\infty \frac{dt}{(t-\zeta)(t+1)} = \frac{1}{1+\zeta}\left\{\int_0^\infty \frac{dt}{t-\zeta} - \int_0^\infty \frac{dt}{t+1}\right\}$$

$$= \frac{-\ln(-\zeta)}{\zeta+1} \sim c \ln\zeta + O(\ln\zeta)\ .$$

(3) $f(t) = \ln t + O(\ln t),\ t \to 0.$

$$I \sim \int_0^\infty \frac{\ln (t-\zeta)}{(t-\zeta)(t+1)}\, dt = \frac{1}{2} \int_0^\infty \frac{d \ln^2(t-\zeta)}{t+1} = c \ln^2\zeta + O(\ln^2\zeta) \quad .$$

BIBLIOGRAPHY

[1] Stoker, J.J. and Peters, A. S., The Motion of a Ship, as a Floating Rigid Body, in a Seaway. IMM-NYU 203, January, 1954.

[2] Peters, A. S., The Effect of a Floating Mat on Water Waves. Communications on Pure and Applied Mathematics, Vol. III, No. 4, December, 1950.

APPROVED DISTRIBUTION LIST

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 438 (3)

Commanding Officer
Office of Naval Research
Branch Office
The John Crerar Library Bldg.
86 East Randolph Street
Chicago 1, Illinois (1)

Commanding Officer
Office of Naval Research
Branch Office
346 Broadway
New York 13, N. Y. (2)

Commanding Officer
Office of Naval Research
Branch Office
1030 East Green Street
Pasadena 1, Calif. (1)

Commanding Officer
Office of Naval Research
Branch Office
1000 Geary Street
San Francisco 24, Calif. (1)

Commanding Officer
Office of Naval Research
Navy No. 100, Fleet Post Office
New York, New York (5)

Director
Naval Research Laboratory
Washington 25, D. C.
Attn: Code 2021 (6)

Documents Service Center
Armed Services Technical
Information Agency
Knott Building
Dayton 2, Ohio (5)

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D. C.
Attn: Research Division (1)

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D. C.
Attn: Research and Development Division (1)

Office of Ordnance Research
Department of the Army
Washington 25, D. C. (1)

Commander
Air Research and Development Comma
Office of Scientific Research
P. O. Box 1395
Baltimore 18, Md.
Attn: Fluid Mechanics Division(1)

Director of Research
National Advisory Committee
for Aeronautics
1724 F Street, Northwest
Washington 25, D. C. (1)

Director
Langley Aeronautical Laboratory
National Advisory Committee
for Aeronautics
Langley Field, Virginia (1)

Director
National Bureau of Standards
Washington 25, D. C.
Attn: Fluid Mechanics Section (1)

Professor R. Courant
Institute of Mathematical
Sciences, N. Y. U.
25 Waverly Place
New York 3, N. Y. (1)

Professor G. Kuerti
Department of Mechanical
Engineering
Case Institute of Technology
Cleveland, Ohio (1)

Professor W.R. Sears, Director
Graduate School of Aeronautical
Engineering
Cornell University
Ithaca, New York (1)

Chief, Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Research Division (1)
Code 420 Preliminary Design (1)

Commander
Naval Ordnance Test Station
3202 E. Foothill Blvd.
Pasadena, California (1)

Commanding Officer and Director
David Taylor Model Basin
Washington 7, D. C.
Attn: Hydromechanics Lab. (1)
Hydrodynamics Div. (1)
Library (1)

California Institute of Technology
Hydrodynamic Laboratory
Pasadena 4, California (1)

Professor A.T. Ippen
Hydrodynamics Laboratory
Massachusetts Institute of Technology
Cambridge 39, Mass. (1)

Dr. Hunter Rouse, Director
Iowa Institute of Hydraulic Research
State University of Iowa
Iowa City, Iowa (1)

Stevens Institute of Technology
Experimental Towing Tank
711 Hudson Street
Hoboken, New Jersey (1)

Dr. L. G. Straub
St. Anthony Falls Hydraulic Laboratory
University of Minnesota
Minneapolis 14, Minn. (1)

Dr. G. H. Hickox
Engineering Experiment Station
University of Tennessee
Knoxville, Tennessee (1)

Mr. C. A. Gongwer
Aerojet General Corporation
6352 N. Irwindale Avenue
Azusa, California (1)

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 416 (1)
Code 460 (1)

Chief, Bureau of Yards and Docks
Department of the Navy
Washington 25, D. C.
Attn: Research Division (1)

Commanding Officer and Director
David Taylor Model Basin
Washington 7, D. C.
Attn: Ship Division (1)

Hydrographer
Department of the Navy
Washington 25, D. C. (1)

Director
Waterways Experiment Station
Box 631
Vicksburg, Mississippi (1)

Office of the Chief of Engineers
Department of the Army
Gravelly Point
Washington 25, D. C. (1)

Beach Erosion Board
U.S. Army Corps of Engineers
Washington 25, D. C. (1)

Commissioner, Bureau of Reclamation
Washington 25, D. C. (1)

Dr. G. H. Keulegan
National Hydraulic Laboratory
National Bureau of Standards
Washington 25, D. C. (1)

Brown University
Graduate Division of Applied Mathematics
Providence 12, Rhode Island (1)

California Institute of Technology
Hydrodynamics Laboratory
Pasadena 4, California
Attn: Professor M.S. Plesset (1)
Professor V.A. Vanoni (1)

DISTRIBUTION LIST (CONT.)

Professor M. L. Albertson
Department of Civil Engineering
Colorado A. + M. College
Fort Collins, Colorado (1)

Professor G. Birkhoff
Department of Mathematics
Harvard University
Cambridge 38, Mass. (1)

Massachusetts Institute of
Technology
Department of Naval Architecture
Cambridge 39, Mass. (1)

Dr. R. R. Revelle
Scripps Institute of Oceanography
La Jolla, California (1)

Stanford University
Applied Mathematics and
Statistics Laboratory
Stanford, Calif. (1)

Professor J.W. Johnson
Fluid Mechanics Laboratory
University of California
Berkeley 4, Calif. (1)

Professor H. A. Einstein
Department of Engineering
University of California
Berkeley 4, Calif. (1)

Dean K. E. Schoenherr
College of Engineering
University of Notre Dame
Notre Dame, Indiana (1)

Director
Woods Hole Oceanographic
Institute
Woods Hole, Mass. (1)

Hydraulics Laboratory
Michigan State College
East Lansing, Mich.
Attn: Professor H.R. Henry (1)

Date Due

	PRINTED	IN U. S. A.	

NYU
IMM
227 Spinelli
AUTHOR
A potential problem con-
TITLE nected with an infinite
dock in a running stream

NYU
IMM
227
Spinelli

A potential problem connected
with an infinite dock in a
running stream.